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On the Index of Matrices and Singular Matrix Pencils

The relations between the Kronecker structure of a reell, singular pencil $sE - A$ and the index of the rational matrix $(sE - A)^D E$, where $(sE - A)^D$ denotes the Drazin inverse of $sE - A$, are investigated.

1. Introduction

The index of linear time-invariant differential-algebraic equations (DAE)

$$E\,\dot{x}(t) = A\,x(t) + f(t) \quad \text{(time domain)} \quad \text{or} \quad (sE - A)\,X(s) = E\,x(t_0) + F(s) \quad \text{(frequency domain)}$$

is defined as the index of the associated matrix pencil $sE - A$ with $E, A \in \mathbb{R}^{l \times n}$. The numerical properties of DAE solvers depend essentially on the index of the matrix $(sE - A)^{-1}E$. The index of the pencil $sE - A$ equals the index of the matrix $(sE - A)^{-1}E$ in case of regular matrix pencils. As for singular matrix pencils, the inverse $(sE - A)^{-1}$ has to be replaced by a suitable generalized inverse. The relations between the Kronecker structure of a singular pencil $sE - A$ and the index of $(sE - A)^D E$, where $(sE - A)^D$ denotes the Drazin inverse of $sE - A$, are investigated.

2. The Index of Matrices and Matrix Pencils

Let $M \in \mathbb{F}^{n \times n}$ denote an $n \times n$-matrix over a field $\mathbb{F}$. Then there exists a regular matrix $T \in GL(n, \mathbb{F})$ such that

$$T\,M\,T^{-1} = \begin{pmatrix} C & 0 \\ 0 & N \end{pmatrix}, \tag{1}$$

where C is a regular matrix and $N = \operatorname{diag}(N_1, \ldots, N_d)$ is a nilpotent block-diagonal matrix consisting of $\nu_i \times \nu_i$-dimensional Jordan blocks associated with the eigenvalue zero. The *index of the matrix* M is defined as the size of the greatest Jordan block N_i, i.e., $\operatorname{ind}(M) := \max\{\nu_1, \ldots, \nu_d, 0\}$. The index can be computed as follows: $\operatorname{ind}(M) = \min\{i \in \mathbb{N} : \operatorname{rank} M^i = \operatorname{rank} M^{i+1}\}$.

Let be $E, A \in \mathbb{R}^{l \times n}$. A matrix pencil $sE - A$ is said to be *regular* if $l = n$ and $\det(sE - A) \not\equiv 0$. Otherwise the matrix pencil is called *singular*. Each regular matrix pencil can be transformed into *Weierstrass canonical form* [1]

$$P\,(sE - A)\,Q = \begin{pmatrix} sI - W & 0 \\ 0 & sN - I \end{pmatrix}, \qquad P, Q \in GL(n, \mathbb{R}). \tag{2}$$

The matrix W is a square matrix, the identity matrix is denoted by I and the matrix N is a nilpotent matrix. The *index of a regular matrix pencil* $sE - A$ is defined by $\operatorname{ind}(E, A) := \operatorname{ind}(N)$. Using the concept of the normal form one can obtain an expression for the index of the matrix pencil in terms of the original system matrices. Let be $s \in \mathbb{C}$ and $\det(sE - A) \neq 0$. Then $\operatorname{ind}(E, A) = \operatorname{ind}((sE - A)^{-1}E)$. As for regular matrix pencils, the condition $\det(sE - A) \neq 0$ is fulfilled for all s belonging to an open and dense subset of $\mathbb{C}$. This implies that $\operatorname{ind}(E, A) = \operatorname{ind}((sE - A)^{-1}E)$ remains true even if s is not fixed at a particular value and the index of $\operatorname{ind}((sE - A)^{-1}E)$ is computed over the field $\mathbb{R}(s)$ of rational functions in s with reell coefficients.

Each singular matrix pencil $sE - A$ can be transformed into *Kronecker canonical form* (cf. [1])

$$P\,(sE - A)\,Q = \operatorname{diag}(sE_r - A_r, L_{\varepsilon_1}, \ldots, L_{\varepsilon_p}, L_{\eta_1}^{\top}, \ldots, L_{\eta_q}^{\top}), \qquad P \in GL(l, \mathbb{R}), \quad Q \in GL(n, \mathbb{R}), \tag{3}$$

where $sE_r - A_r$ denotes a regular pencil and each block L_k is a $k \times (k+1)$-dimensional pencil of the form $L_k = (s\delta_{i,j} + \delta_{i,j-1})_{ij}$. The nonnegative integers ε_i and η_j are called *right* and *left Kronecker indices*, respectively. We define the *index of a singular matrix pencil* $sE - A$ as the index of the regular part: $\operatorname{ind}(E, A) := \operatorname{ind}(E_r, A_r)$.

Let be $M \in \mathbb{F}^{n \times n}$ and $k := \operatorname{ind}(M)$. A matrix $D \in \mathbb{F}^{n \times n}$ that fulfills $D = DAD$, $AD = DA$ and $A^{k+1}D = A^k$ is called *Drazin inverse of* M and will be denoted by M^D [2]. If $M = T^{-1}\operatorname{diag}(C, N)\,T$ (see Eq. (1)) then $M^D = T^{-1}\operatorname{diag}(C^{-1}, 0)\,T$. The matrix M^D is a generalized inverse of M if and only if $\operatorname{ind}(M) \leq 1$ (see [3]).

3. The Index of the Rational Matrix $(sE - A)^D E$

The following example shows that — in contrast to regular matrix pencils — the index of the rational matrix $(sE - A)^D E$ is not uniquely determined by the index of the matrix pencil:

$$
\begin{aligned}
sE_1 - A_1 &= \left(\begin{array}{cc|c} 1 & s & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), & \operatorname{ind}((sE_1 - A_1)^D E_1) &= \operatorname{ind}\left(\begin{array}{cc|c} 0 & -s & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) = 2 \\
sE_2 - A_2 &= \left(\begin{array}{c|cc} 0 & 1 & s \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \end{array}\right), & \operatorname{ind}((sE_2 - A_2)^D E_2) &= \operatorname{ind}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 1
\end{aligned}
$$

Both matrix pencils $sE_1 - A_1$ and $sE_2 - A_2$ have the same index 2, but the indices of the associated rational matrices $(sE_i - A_i)^D E_i$ $(i = 1, 2)$ do not coincide. Even in case of $\operatorname{ind}(sE - A) = 1$ the index of $(sE - A)^D E$ is not uniquely determined by the Kronecker structure of $sE - A$. In the next example, each matrix pencil $sE_3 - A_3$ and $sE_4 - A_4$ consists of a L_2- and a $L_0^\top$-block only. However, the rational matrices $(sE_i - A_i)^D E_i$ $(i = 3, 4)$ have different indices.

$$
\begin{aligned}
sE_3 - A_3 &= \left(\begin{array}{ccc} s & 1 & 0 \\ 0 & s & 1 \\ \hline 0 & 0 & 0 \end{array}\right), & \operatorname{ind}((sE_3 - A_3)^D E_3) &= \operatorname{ind}\left(\begin{array}{cc|c} \frac{1}{s} & -\frac{1}{s^2} & 0 \\ 0 & \frac{1}{s} & 0 \\ \hline 0 & 0 & 0 \end{array}\right) = 1 \\
sE_4 - A_4 &= \left(\begin{array}{ccc} 1 & s & 0 \\ 0 & 1 & s \\ \hline 0 & 0 & 0 \end{array}\right), & \operatorname{ind}((sE_4 - A_4)^D E_4) &= \operatorname{ind}\begin{pmatrix} 0 & 1 & -s \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = 3
\end{aligned}
$$

The following theorem gives an upper bound for the index of the rational matrix $(sE - A)^D E$. We restrict ourselves to $\operatorname{ind}(sE - A) = 1$ (over $\mathbb{R}(s)$) where the Drazin inverse is a generalized inverse.

Theorem 1. *Let be* $\operatorname{ind}(sE - A) = 1$ *and* $k := \operatorname{ind}(E, A)$. *Futhermore, let* ε_i *and* η_j *denote the right and left Kronecker indices, respectively. Then there holds the inequality*

$$k \leq \operatorname{ind}((sE - A)^D E) \leq \max_{i,j} \{k, \varepsilon_i + 1, \eta_j\},$$

where the index $\operatorname{ind}(\cdot)$ *has to be calcutated over the field* $\mathbb{R}(s)$.

Proof. Let be $s \in \mathbb{R}$. For a moment, we will replace the matrices E and A by linear operators $\mathcal{E}$ and $\mathcal{A}$. On the image $\mathcal{U} := \operatorname{im}(s\mathcal{E} - \mathcal{A})^D$ the Drazin inverse operator $(s\mathcal{E} - \mathcal{A})^D$ acts as a (normal) inverse operator of $s\mathcal{E} - \mathcal{A}$ [3]. Because of $\operatorname{ind}(s\mathcal{E} - \mathcal{A}) = 1$ the vector space $\mathcal{U}$ coincides with the image $\operatorname{im}(s\mathcal{E} - \mathcal{A})$. Hence, the operator pencil $s\mathcal{E} - \mathcal{A}$ becomes regular (i.e., invertible) on the subspace $\mathcal{U}$ and we have $\operatorname{ind}(s\mathcal{E} - \mathcal{A}) = \operatorname{ind}((s\mathcal{E} - \mathcal{A})^{-1}\mathcal{E}) = \operatorname{ind}((s\mathcal{E} - \mathcal{A})^D \mathcal{E})$ for the restriction of the operators $\mathcal{E}$ and $\mathcal{A}$ to the subspace $\mathcal{U}$.

Next, we have to consider what the restriction to $\operatorname{im}(sE - A)$ (and the inversion on this subspace) means in terms of the matrix pencil $sE - A$. The image of $sE - A$ can be written as a direct sum

$$\operatorname{im}(sE - A) = P^{-1}\left[\operatorname{im}(sE_r - A_r) \oplus \operatorname{im} L_{\varepsilon_1} \oplus \cdots \oplus \operatorname{im} L_{\varepsilon_p} \oplus \operatorname{im} L_{\eta_1}^\top \oplus \cdots \oplus \operatorname{im} L_{\eta_q}^\top\right] Q^{-1}$$

associated with the block-diagonal representation of $sE - A$ (see Eq. (3)). The regular part can be inverted as usual, i.e., the Jordan block structure of the nilpotent matrix N (see Eq. (2)) is sustained in $(sE - A)^D E$. Now, let us consider the singular part. Note that $p = q$ because the matrix pencil is assumed to be square. For each L_{ε_i}-block the matrix $(sE - A)^D E$ is a linear mapping between two $(\varepsilon_i + 1)$-dimensional subspaces. Hence, the index of this linear mapping can not exceed the value $\varepsilon_i + 1$. In case of a $L_{\eta_j}^\top$-block the matrix $(sE - A)^D E$ is a linear mapping between two η_j-dimensional subspaces. Obviously, the index associated with this block can not be greater than η_j. This completes the proof.

4. References

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