REINSCHKE, K. J.; RÖBENACK, K.

Graph-theoretic characterization of structural controllability for singular differential-algebraic equations

Linear time-invariant differential-algebraic equations (DAEs) $E\dot{x} = Ax + Bu$, where the matrix pencil $sE - A$ is possibly singular, are investigated. The sparsity pattern of the matrices E, A and B determines the structure of a bipartite graph. A graph-theoretic condition for structural controllability is derived and an example shows the use of this condition.

1. Singular Systems and Controllability

Many physical systems can be modelled by *differential-algebraic equations (DAE)* of the form

$$E\,\dot{x}(t) = A\,x(t) + B\,u(t), \qquad E, A \in \mathbb{R}^{l\times n}, \quad B \in \mathbb{R}^{l\times m}. \tag{1}$$

The DAE is said to be *regular* if the associated matrix pencil $sE - A$ is regular, i.e., if $l = n$ and $\det(sE - A) \not\equiv 0$. Otherwise both the DAE and the matrix pencil are called *singular*. Regular DAE have been investigated by many authors (cf. [1]). In this contribution we will consider the more general case of singular DAE. Singular DAEs appear in many applications. For example, singular DAEs arise if we require the solution of the state-space system $\dot{x} = Ax + Bu$, $y = Cx$ to satisfy $0 = Fy\,(= FCx)$. The matrix pencil $\binom{sI-A}{FC}$ of the combined system is rectangular and therefore singular. Another reason for studying singular DAEs is because the class of regular systems is not closed under the action of a feedback group, i.e., the property of regularity is not feedback invariant [2].

Each matrix pencil $sE - A$ can be transformed into a special block-diagonal form called *Kronecker canonical form*

$$P\,(sE - A)\,Q = \mathrm{diag}\,(sI - W, sN - I, sE_\varepsilon - A_\varepsilon, sE_\eta - A_\eta), \qquad P \in GL\,(l, \mathbb{R}),\ Q \in GL(n, \mathbb{R}),$$

where I, W and N denote an identity matrix, a square matrix and a nilpotent matrix, respectively [3]. The roots of $\det(sI - W) = 0$ are called *finite zeros*. The blocks $sE_\varepsilon - A_\varepsilon = \mathrm{diag}\,(L_1, \ldots, L_p)$ and $sE_\eta - A_\eta = \mathrm{diag}\,(L_1^\top, \ldots, L_q^\top)$ are block-diagonal matrices consisting of $k \times (k+1)$-dimensional blocks $L_k = (s\delta_{i,j} + \delta_{i,j-1})_{ij}$.

In the frequency domain, regular DAEs are solvable for *all* input signals $U \in \mathbb{R}(s)^m$, where $\mathbb{R}(s)$ denotes the field of rational functions. Singular DAE are only solvable for *admissible* input signals, i.e., for functions $U \in \mathbb{R}(s)^m$ that fulfill the condition $\mathrm{rank}\,(sE - A) = \mathrm{rank}\,(sE - A, B\,U(s))$ for all s belonging to an open and dense subset of $\mathbb{C}$.

Following [1] a regular system is called *controllable* if the matrix pencil $(sE - A, B)$ has full row rank for all $s \in \mathbb{C}$. Hence, a regular DAE is controllable if and only if the $n \times (n+m)$-dimensional pencil $(sE - A, B)$ has no finite zeros, i.e., there are regular matrices $\tilde{P} \in GL(n, \mathbb{R})$ and $\tilde{Q} \in GL(n+m, \mathbb{R})$ such that $\tilde{P}\,(sE - A, B)\,\tilde{Q} = \mathrm{diag}\,(s\tilde{N} - I, s\tilde{E}_\varepsilon - \tilde{A}_\varepsilon)$.

We proceed to define a notion of controllability for singular DAEs. Evidently, the regular part, i.e., $\mathrm{diag}\,(sI - W, sN - I)$ and the associated rows of the transformed input matrix PB should be controllable. Additional columns of the input matrix associated with the singular block $sE_\eta - A_\eta$ may increase the number of finite zeros. The newly generated regular parts should be controllable by the remaining columns of the input matrix. The block $sE_\varepsilon - A_\varepsilon$ does not play an essential role since this block is row regular for all $s \in \mathbb{C}$.

Definition 1. *A (possibly singular) DAE (1) is said to be* controllable *if the associated matrix pencil $(sE - A, B)$ has no finite zeros.*

As for singular DAEs, controllability according to Definition 1 means that there exists regular matrices $\tilde{P}$ and $\tilde{Q}$ such that $\tilde{P}\,(sE - A, B)\,\tilde{Q} = \mathrm{diag}\,(s\tilde{N} - I, s\tilde{E}_\varepsilon - \tilde{A}_\varepsilon, s\tilde{E}_\eta - \tilde{A}_\eta)$. The absence of finite zeros can be checked by rank computations: Let $r := \max\{\mathrm{rank}\,(sE - A, B) \text{ for } s \in \mathbb{C}\}$ denote the *normal rank* of $(sE - A, B)$. Then the DAE system is controllable if and only if

$$\forall s \in \mathbb{C}: \quad \mathrm{rank}\,(sE - A, B) = r. \tag{2}$$

2. Graph-theoretic Interpretation

A (Boolean) structure matrix $[M]$ is a matrix whose entries are either fixed at zero (denoted by "0") or indeterminate values (denoted by "×"). Fixing all indeterminate entries of $[M]$ at some particular values we obtain an *admissible realization* $M \in [M]$. Consider a structure matrix $[M]$ with h non-zero entries. The set of admissible realizations is isomorphic to the real vector space $\mathbb{R}^h$. We say "a property holds *structurally* for $[M]$" if the property under consideration is met for all $M \in [M]$ belonging to an open and dense subset of $\mathbb{R}^h$.

Any $l \times (n+m)$ pencil $[sE - A, B]$ of structure matrices can be represented by a bipartite graph $G([sE-A,B])$ whose vertex set consists of l U-vertices $u_1, \ldots, u_l$ and $n+m$ V-vertices $v_1, \ldots, v_{n+m}$. There is a thin edge or a bold edge between the vertices u_i and v_j if $e_{ij} \not\equiv 0$ or $a_{ij} \not\equiv 0$, respectively. Furthermore, there is thin edge between u_i and v_{n+j} if $b_{ij} \not\equiv 0$. A subset of edges is called a *matching* if any two edges of it do not have a common vertex.

Theorem 1. *Let r denote the maximum cardinality of a matching within the bipartite graph $G([sE-A,B])$. The associated class of DAE is structurally controllable if and only if*

a) the graph $G([sE-A,B])$ has a matching of cardinality r without bold edges, and
b) the consistent parts of the Dulmage-Mendelsohn-decomposition of $G([sE-A,B])$ do not contain any bold edges.

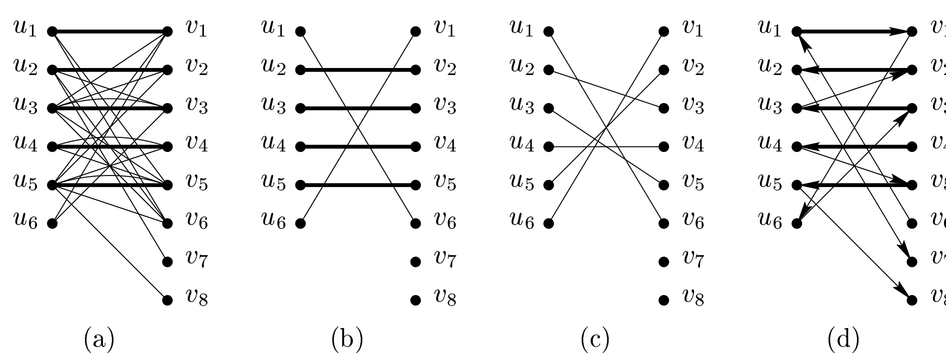
Proof. The maximum cardinality r of a matching within $G([sE-A,B])$ coincides with the normal rank of $(sE - A, B)$ for almost all $(E, A, B) \in [E, A, B]$. Structurally, the existence of a matching of cardinality r that does not contain any bold edge is equivalent to (2) for $s = 0$. Futhermore, Condition (2) holds structurally for all $s \neq 0$ if and only if the consistent parts of the Dulmage-Mendelsohn-decomposition do not contain any bold edges (see [4]).

Example 1. The solution of the state-space system shown in [5] has to fulfill one additional constraint. The system can be interpreted as a singular DAE with the following structure matrices ($l = 6$, $n = 5$, $m = 3$):

$$[E] = \begin{pmatrix} \times & 0 & 0 & 0 & 0 \\ 0 & \times & 0 & 0 & 0 \\ 0 & 0 & \times & 0 & 0 \\ 0 & 0 & 0 & \times & 0 \\ 0 & 0 & 0 & 0 & \times \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad [A] = \begin{pmatrix} 0 & 0 & 0 & 0 & \times \\ 0 & 0 & \times & 0 & 0 \\ \times & \times & \times & 0 & \times \\ 0 & 0 & 0 & \times & \times \\ \times & \times & 0 & \times & \times \\ \times & 0 & \times & 0 & 0 \end{pmatrix}, \quad [B] = \begin{pmatrix} \times & 0 & 0 \\ \times & \times & 0 \\ \times & 0 & 0 \\ 0 & 0 & 0 \\ \times & 0 & \times \\ 0 & 0 & 0 \end{pmatrix}$$

The associated bipartite graph $G([sE-A,B])$ is shown in Fig. 1(a). A matching of maximum cardinality $r = l = 2$ can be found (see Fig. 1(b)). Moreover, the graph $G([sE-A,B])$ contains a matching of the same cardinality without bold edges (Fig. 1(c)). Fig. 1(d) shows that the Dulmage-Mendelsohn-decomposition does not have a consistent part. Hence, the DAE is structural controllable.

Figure 1: Bipartite graph $G([sE-A,B])$



3. References

1 Yip, E. L.; Sincovec, R. F.: Solvability, Controllability, and Observability of Continous Descriptor Systems. IEEE Trans. **AC-26** (1981), 702–707.
2 Özçaldiran, K.; Lewis, F. L.: On the Regularizability of Singular Systems. IEEE Trans. **AC-35** (1990), 1156–1160.
3 Gantmacher, F. R.: Theory of Matrices II. Chelsea, New York, 1959.
4 Murota, K.: Systems Analysis by Graphs and Matroids: Structural Solvability and Controllability. Springer-Verlag, 1987.
5 Konigorski, U.: Entwurf linearer Verkopplungsregelungen. Workshop des GMA-Fachausschusses 1.4 "Theoretische Verfahren der Regelungstechnik", Interlaken (Schweiz), 28.09.-01.10.1997, Vortragsunterlagen, 111–120.

Addresses: Prof. Dr. K. J. Reinschke, Dipl.-Ing. K. Röbenack, Technische Universität Dresden, Institut für Regelungs- und Steuerungstheorie, Mommsenstr. 13, 01062 Dresden, Germany.